<u>Proposition 1</u> (Polyakov-Wiegmann formula):

Let $f, g : \Sigma \longrightarrow G$ be smooth maps.

Then

$$\exp\left(-S_\Sigma(fg)\right)$$
$$= \exp\left(-S_\Sigma(f) - S_\Sigma(g) - \frac{\sqrt{-1}\,k}{2\pi}\int_\Sigma \mathrm{Tr}\left(f^{-1}\bar\partial f \wedge \partial g\, g^{-1}\right)\right)$$

holds.

<u>Proof</u>:

Note that $\mathrm{Tr}(\omega\wedge\eta) = (-1)^{pq}\,\mathrm{Tr}(\eta\wedge\omega)$ for differential forms ω and η of degrees p and q respectively. Then we compute

$$I = \int_\Sigma \mathrm{Tr}\left(\left((fg)^{-1}\partial(fg)\right)\wedge (fg)^{-1}\bar\partial(fg)\right)$$

$$= \int_\Sigma \mathrm{Tr}\left(f^{-1}\partial f\wedge f^{-1}\bar\partial f + g^{-1}\partial g\wedge g^{-1}\bar\partial g\right)$$

$$+ \int_\Sigma \mathrm{Tr}\left(f^{-1}\partial f\wedge\bar\partial g\, g^{-1} + \partial g\, g^{-1}\wedge f^{-1}\bar\partial f\right)$$

(exercise)

Applying $df^{-1} = -f^{-1} df\, f^{-1}$ and Stokes' theorem gives the result (exercise). $\square$

Set
$$T_\Sigma(f,g) = -\frac{\sqrt{-1}\,K}{2\pi}\left(\int_\Sigma \mathrm{Tr}\left(f^{-1}\bar{\partial} f \wedge \partial g\, g^{-1}\right)\right)$$

For the complexification $G_{\mathbb{C}} = SL(2,\mathbb{C})$ consider
$$f : D \longrightarrow G_{\mathbb{C}} \quad \text{where } D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$$

Denote the complement in $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ by D_∞, i.e. $D_\infty = \{ z \in \mathbb{C} \mid |z| \geq 1 \} \cup \{\infty\}$

Consider $f : \mathbb{CP}^1 \longrightarrow G_{\mathbb{C}}$ smooth s.t.
$$f|_D = f_0 \quad \text{and} \quad f|_{D_\infty} = f_\infty$$

for some f_∞. Then $\exp(-S_{\mathbb{CP}^1}(f)) \in \mathbb{C}$ depends on extension of f.

Consider second extension $f' : \mathbb{CP}^1 \longrightarrow G_{\mathbb{C}}$.

Then $\quad f' = fh,$

where $h : \mathbb{CP}^1 \longrightarrow G_{\mathbb{C}}$ with $h|_D = e$ (unit of $G_{\mathbb{C}}$) and denote $h_\infty = h|_{D_\infty}$.

Using the Polyakov-Wiegmann formula we then obtain

$$\exp\left(-S_{\mathbb{CP}^1}(fh)\right) = \exp\left(-S_{\mathbb{CP}^1}(f) - S_{\mathbb{CP}^1}(h) + \Gamma_{\mathbb{CP}^1}(f,h)\right)$$

and $\Gamma_{\mathbb{CP}^1}(f,h) = \Gamma_{\infty}(f_\infty, h_\infty)$

$$\Rightarrow \exp\left(-S_{\mathbb{CP}^1}(fh)\right) = \exp\left(-S_{\mathbb{CP}^1}(f) - S_{\mathbb{CP}^1}(h) + \Gamma_{D_\infty}(f_\infty, h_\infty)\right) \quad (*)$$

Denote by $\mathrm{Map}_0(D_\infty, G_{\mathbb{C}})$ the set of smooth maps $\varphi : D_\infty \longrightarrow G_{\mathbb{C}}$ with $\varphi(\infty) = e$ using $z^{-1} = re^{\sqrt{-1}\theta}$ we set $\varphi(re^{\sqrt{-1}\theta}) = p_r(e^{\sqrt{-1}\theta})$.

The map $p_r : S^1 \longrightarrow G_{\mathbb{C}}$, $0 \leq r \leq 1$ defines loop of $G_{\mathbb{C}}$ for fixed r, for $r = 0$

$\longrightarrow p_0 : \theta \longmapsto e$ for $\theta \in S^1$

$\longrightarrow p_r$, $0 \leq r \leq 1$ corresponds to a path in $LG_{\mathbb{C}}$ starting at $e \in LG_{\mathbb{C}}$ (constant map from S^1 to $G_{\mathbb{C}}$)

Introduce equivalence relation $\sim$ on

$$\mathrm{Map}_0(D_\infty, G_\mathbb{C}) \times \mathbb{C}$$

by setting for $(f_\infty, u), (g_\infty, v) \in \mathrm{Map}_0(D_\infty, G_\mathbb{C})$:

$$(f_\infty, u) \sim (g_\infty, v)$$

<u>iff:</u>

a) $f_\infty(z) = g_\infty(z)$ holds for $z \in \partial D_\infty$.

b) for $g_\infty = f_\infty h$ one has

$$v = u \exp\left(-S_{\mathbb{CP}^1}(h) + T_{D_\infty}(f_\infty, h_\infty)\right)$$

$\longrightarrow$ $\mathrm{Map}_0(D_\infty, G_\mathbb{C}) \times \mathbb{C}/\sim$ gives line bundle $\mathcal{L}$ on $LG_\mathbb{C}$

Define projection map $\pi : \mathcal{L} \longrightarrow LG_\mathbb{C}$ by

$$\pi([f_\infty, u]) = f_\infty \circ \iota$$

where $\iota : \partial D \longrightarrow D_\infty$ is inclusion map

In the above, f_∞ and g_∞ correspond to different paths γ_1 and γ_2 s.t.

$$\gamma_i \, [0,1] \longrightarrow LG_\mathbb{C}, \quad i = 1,2$$

$$\gamma_1(0) = \gamma_2(0) = e$$

a) $\Longleftrightarrow$ $\gamma_1(1) = \gamma_2(1)$, b) $\Leftrightarrow$ holonomy along $\gamma_1 \cdot \gamma_2^{-1}$

$\longrightarrow$ analogous to line bundle of Prop. 3 in § 2

<u>one can show</u>: $\mathcal{L}$ is isomorphic to k-fold tensor product of fundamental line bundle.

Suppose $f: \mathbb{CP}^1 \longrightarrow G_{\mathbb{C}}$ is extension of $f_0: D \longrightarrow G_{\mathbb{C}}$.

Define

$$\exp(-S_D(f_0)) \equiv [f_\infty, \exp(-S_{\mathbb{CP}^1}(f))]$$

<u>Lemma</u>:

For $f_0: D \longrightarrow G_{\mathbb{C}}$, $[f_\infty, \exp(-S_{\mathbb{CP}^1}(f))]$ does not depend on choice of extension of f_0
$\longrightarrow$ element in fibre of $\mathcal{L}$ over $f_0 \circ \iota$.

<u>Proof</u>:

Take another extension $f': \mathbb{CP}^1 \longrightarrow G_{\mathbb{C}}$ of $f_0: D \longrightarrow G_{\mathbb{C}}$ with $f' = fh$. Then

$$(f_\infty, \exp(-S_{\mathbb{CP}^1}(f))) \sim (f_\infty h, \exp(-S_{\mathbb{CP}^1}(fh)))$$

follows from Polyakov-Wiegmann formula. $\square$

Dual line bundle $\mathcal{L}^{-1}$:

Denote by $\mathrm{Map}_0(D, G_{\mathbb{C}})$ set of smooth maps

$$\varphi : D \longrightarrow G_{\mathbb{C}} \quad \text{with } \varphi(0) = e$$

and define equiv. relation $(f_0, u) \sim (g_0, v)$ by

a) $f_0(z) = g_0(z)$ for $z \in \partial D$

b) $g_0(z) = f_0 h_0 \longrightarrow v = u \exp(-S_{\mathbb{CP}^1}(h) + T_D(f_0, h_0))$

$\longrightarrow$ Define $\mathcal{L}^{-1}$ as $\mathrm{Map}_0(D, G_{\mathbb{C}}) \times \mathbb{C}/\sim$

$\longrightarrow$ $\exp(-S_{D_\infty}(f_\infty))$ is well-defined as element of fibre of $\mathcal{L}^{-1}$ over $f_\infty \circ \iota$.

Denote by $\gamma : S^1 \longrightarrow G_{\mathbb{C}}$ the loop defined by $f \circ \iota$. Then we have pairing

$$\mathcal{L}_\gamma \times \mathcal{L}_\gamma^{-1} \longrightarrow \mathbb{C}$$

given by

$$\langle [f_\infty, u], [f_0, v] \rangle = uv \exp(S_{\mathbb{CP}^1}(f))$$

where $\mathcal{L}_\gamma$ is the fibre of $\mathcal{L}$ over γ.

$\longrightarrow$ pairing well-defined as right-hand side is independent of representations of equivalence classes.

Definition:

The following operation

$$\exp(-S_D(g_1)) \circ \exp(-S_D(g_2))$$

$$= \exp(-T_D(g_1,g_2)) \exp(-S_D(g_1, g_2)),$$

for $g_i : D \longrightarrow G_{\mathbb{C}}$, $i=1,2$, defines a product

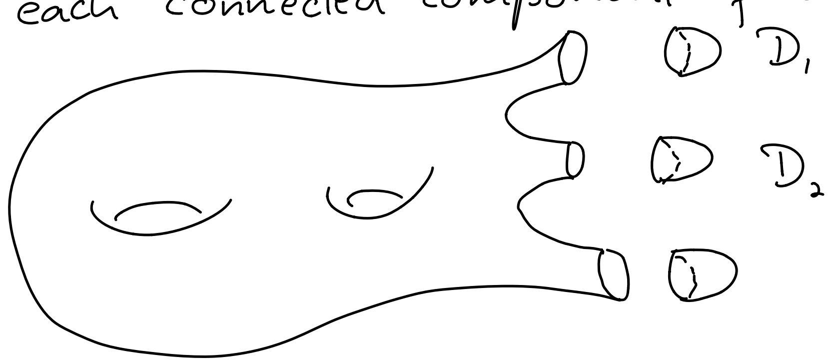
$$\mathcal{L}_{\gamma_1} \times \mathcal{L}_{\gamma_2} \longrightarrow \mathcal{L}_{\gamma_1 \cdot \gamma_2}$$

where $\gamma_i = g \circ \iota$. This product equippes $\widehat{LG_{\mathbb{C}}} = \mathcal{L} \setminus \underset{\substack{\uparrow \\ \text{zero section}}}{s(LG_{\mathbb{C}})}$ with a group structure.

Next, let Σ be compact Riemann surface with boundary. $\partial\Sigma$ is homeomorphic to a disjoint union of circles and we have diffeomorphisms

$$p_i : S^1 \longrightarrow \partial\Sigma, \quad 1 \leq i \leq m$$

for each connected component of $\partial\Sigma$.



Glue the boundary of unit discs D_i, $1 \leq i \leq m$, with $p_i(S^1)$, $1 \leq i \leq m$, to obtain a closed Riemann surface $\widetilde{\Sigma}$.

For a smooth map $g : \Sigma \longrightarrow G_{\mathbb{C}}$ define the extension to $\widetilde{\Sigma}$ as $\widetilde{g} : \widetilde{\Sigma} \longrightarrow G_{\mathbb{C}}$ and the restriction on D_i by g_i.

$\longrightarrow$ $\exp(-S_{D_i}(g_i))$ defines an element of the fibre of $\mathcal{L}^{-1}_{g \circ p_i}$

$\longrightarrow$ Define $\exp(-S_{\Sigma}(g))$ as element of $\bigotimes_{i=1}^{m} \mathcal{L}^{-1}_{g \circ p_i}$ specified by

$$\left\langle \exp(-S_{\Sigma}(g)), \bigotimes_{i=1}^{m} \exp(-S_{D_i}(g_i)) \right\rangle = \exp(-S_{\widetilde{\Sigma}}(\widetilde{g}))$$

By the Polyakov-Wiegmann formula this definition does not depend on choice of extension $\widetilde{g}$.